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A Study of Birth Rate Functions in Population Models

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Abstract— **Periodic change of population of some species was considered to formulate a single-species population models for different birth rate functions. We used impulsive differential equations where increase occurs in a single pulse once per time period. Considering some assumptions, we derived models for a selection of birth functions. Using a stability analysis we acquired conditions for the stability of the dynamics of these models. The dependence with the bifurcation parameter was also included. These systems generate series of period-doubling bifurcations and dynamics directed to chaotic attractors. A small change in bifurcation parameter or initial values can prominently change the dynamic behaviours of the system. The numerical results show that chaos is followed by period-doubling bifurcations.**

Keywords- bifurcation; chaos, impulsive differential equations; Stage structured population model

I. INTRODUCTION

Recently structured population models have become a dominant modeling formalism in biology. This is not only because they are much simpler than the models governed by partial differential equations but also they can exhibit phenomena similar to those of partial differential models, and many important physiological parameters can be incorporated [1].

In this paper we formulated a stage structured single species population models with different annual birth pulse. In these models we used impulsive differential equations which give the system a mixed nature of both continuous and discrete and these models are subject to short-term perturbations which are often assumed to be in the form of impulses in the modeling process [2]. Some impulsive equations have been introduced in population dynamics in relation to vaccination [3], and chemotherapeutic [4] treatment of disease. We incorporate different birth rate functions of population density to analysis the long term dynamics of the system. For example Hussel model for most frequently used for study of insect population with three free parameters chosen to match the data for a population, and also we consider Shepherd function as birth rate function which is a unity of Ricker, Cushing and Beverton-Holt functions used in fisheries[5].

The paper presents an analytical and numerical study of long term dynamics of the solutions of these periodically forced systems or periodically pulsed systems. The population in the pulsed birth time is characterized by the existence and the stability of equilibria, by the bifurcations that occur when stability is lost and the patterns of dynamics that follow the bifurcation.

In our results and discussions section we present the bifurcation diagram which illustrates the interesting complex dynamic behavior of the models. They reveal that birth pulse in effect, provides a natural period or cyclicity that allows for a period-doubling route to chaos.

II. METHODOLOGY

In this section, we set out the models with different birth rate functions that we investigate, and discuss basic properties. The population size, which is denoted by is divided into immature and mature classes, with the size of each class and respectively. Only the mature population can reproduce. These leads to the following single-species population model with stage structure,

$$\dot{x}(t) = -dx(t) - \delta x(t) \quad (1)$$
$$\dot{y}(t) = \delta x(t) - dy(t) t \in (m, m+1],\ m \in \mathbb{Z}(2)$$

The maturity rate δ>0 and the death rate d >0 are constants, and B(P) is the birth rate function. Since in many species such as insects, fish and many large mammal populations, births are seasonal or occur in regular pulses reproduction takes place in a relatively short period each year [6], we consider impulsive differential equations introducing periodic birth pulses.

$$x(m^+) = x(m^-) + \frac{a}{\left[1+b\left(x(m^-)+y(m^-)\right)\right]^c}\ y(m^-) m \in \mathbb{Z} \quad \text{with}$$
$$a, b, c > 0\ and\ a > d \quad (3)$$

$$x(m^+) = x(m^-) + \frac{k}{1 + q\left(x(m^-) + y(m^-)\right)^{\frac{1}{n}}} y(m^-)$$
$$\text{with} \quad q, n > 0 \quad and\ k > d \quad (4)$$

$$x(m^+) = x(m^-) + \left[\frac{A}{x(m^-) + y(m^-)} + z\right] y(m^-)$$
with $A > 0, d > z > 0$ (5)

$$x(m^+) = x(m^-) + q\big(y(m^-)\big)e^{-(x(m^-)+y(m^-))}$$
with $q > d$ (6)

$$x(m^+) = x(m^-) + \frac{c\big(y(m^-)\big)}{\left(1 + \left[\frac{x(m^-) + y(m^-)}{k}\right]^2\right)}$$
with $k > 0, c > d$ (7)

$x(m^+)$ is the quantity of population after the birth pulse and $x(m^+) = \lim_{m \to m^+} x(m)$ .We have incorporated birth rate functions with the properties that the inverse of birth-rate function exists and that the birth rate when population tends to zero is greater than the death rate and also less than it when population tends to infinity. These pulses reflect biology such as insect, fish populations. In the following section, we will investigate the dynamics of models to analyze period-one solutions, period-doubling bifurcations and chaos of this system. We integrate and solve for the immature population in equation (1) between pulses,

$$x(t) = x_m e^{-(\delta+d)(t-m)}\,, \quad m < t < m+1$$

where x_m the initial population of immatures at time m. Moreover, by adding the equations (1) and (2) of system and the integration between pulses yields,

$$x(t) + y(t) = (x_m + y_m)e^{-d(t-m)}\,, \quad m < t < m+1$$

where y_m is the initial population of matures at time *m*. Hence we get $y(t) = e^{-d(t-m)}\left[y_m + x_m\left(1 - e^{-\delta(t-m)}\right)\right]$ between pulses. At each successive pulse, more of the immature population is added for each case discussed above, the equations ((3)-(7)) yield

$$x_{m+1} = x_m e^{-(d+\delta)} + \frac{ae^{-d}\left[y_m + x_m\left(1 - e^{-\delta}\right)\right]}{[1 + be^{-d}(x_m + y_m)]^c}$$

$$x_{m+1} = x_m e^{-(d+\delta)} + \frac{ke^{-d}\left[y_m + x_m\left(1 - e^{-\delta}\right)\right]}{1 + q[e^{-d}(x_m + y_m)]^{\frac{1}{n}}}$$

$$x_{m+1} = x_m e^{-(d+\delta)} + \left[\frac{A}{e^{-d}(x_m + y_m)} + z\right]\left[y_m + x_m\left(1 - e^{-\delta}\right)\right]e^{-d}$$

$$x_{m+1} = x_m e^{-(d+\delta)} + q\left[y_m + x_m\left(1 - e^{-\delta}\right)\right](e^{-d})\left[e^{-e^{-d}(y_m + x_m)}\right]$$

$$x_{m+1} = x_m e^{-(d+\delta)} + \frac{ce^{-d}\left[y_m + x_m\left(1 - e^{-\delta}\right)\right]}{1 + \left[\frac{e^{-d}(y_m + x_m)}{k}\right]^2}$$

These difference equations describe numbers of immature population and mature population at $t = m + 1$ in terms of values at $t = m$ and the dynamics of each system is determined by the dynamical behavior of model. Thus, in the following we will focus our attention on systems and investigate the various dynamical behaviours. The dynamics of these nonlinear models can be studied as a function of any of the parameters. The above five systems lead to a trivial equilibria at(0,0)and a unique positive equilibria $E^*(x^*, y^*)$ as listed in Table 1. When (0,0) becomes unstable it permits the colonization of the population.

TABLE I**.** Nontrivial equilibria of the five models with birth pulses.

Model	**Equilibria**	**For the globally asyptotical condition is U > 1 for U=**
(3)	$x^* = \frac{a}{b(d+\delta)}\left(U^{\frac{1}{c}} - 1\right)$ $y^* = \frac{a}{b(d+\delta)}\left(U^{\frac{1}{c}} - 1\right)$	$\frac{a\delta}{d(d+\delta)}$
(4)	$x^* = \frac{}{(d+\delta)}\left(\frac{U q - 1}{}\right)^n$ $y^* = \frac{}{b(d+\delta)}\left(\frac{}{q}\right)$	$\frac{k\delta}{d(d+\delta)}$
(5)	$x^* = \frac{A}{z(d+\delta)}\left[\frac{1}{U} - 1\right]^{-1}$ $y^* = \frac{}{z(d+\delta)}\left[\frac{1}{U} - 1\right]$	$\frac{z\delta}{d(d+\delta)}$
(6)	$x^* = \frac{}{(d+\delta)}(ln\mathrm{U})$ $y^* = \frac{}{(d+\delta)}(ln\mathrm{U})$	$\frac{q\delta}{d(d+\delta)}$
(7)	$x^* = \frac{}{\delta(d+\delta)}\sqrt{U - 1}$ $y^* = \frac{}{(d+\delta)}$	$\frac{c\delta}{d(d+\delta)}$

In the linearization of the system as

$$\begin{pmatrix} x_{m+1} \\ y_{m+1} \end{pmatrix} = H \begin{pmatrix} x_m \\ y_m \end{pmatrix}$$

With H equals to the linear counterpart and (0,0) is stable when the eigen values of H are less than one in magnitude. With H defined in each model it can be shown that eigenvalues are less than one if the conditions,

$a < \frac{(1-e^{-d})(1-e^{-(d+\delta)})}{e^{-d}(1-e^{-\delta})} \equiv a_0$, $k < \frac{(1-e^{-d})(1-e^{-(d+\delta)})}{e^{-d}(1-e^{-\delta})} \equiv k_0$, $c < \frac{(1-e^{-d})(1-e^{-(d+\delta)})}{e^{-2d}(1-e^{-\delta})} \equiv c_0$, $q < \frac{(1-e^{-d})(1-e^{-(d+\delta)})}{e^{-d}(1-e^{-\delta})} \equiv q_0$, $c < \frac{(1-e^{-d})(1-e^{-(d+\delta)})}{e^{-d}(1-e^{-\delta})} \equiv c_0$ hold.

In order for a small population to increase, above inequalities should reverse. We can hence list out the following conditions as shown in table 2, so that there exists a unique positive equilibria and exchange of stability is via a transcritical bifurcation.

TABLE2.. Nontrivial equilibria of the two models with birth pulses.

Functions	**Equilibria**	**The average number of offspring that an individual produces over the course of its lifetime, V>1. Here V=**
(3)	$\bar{x}^* = \frac{(1-e^{-d})\left(V^{\frac{1}{c}}-1\right)}{be^{-d}(1-e^{-(d+\delta)})}$ $\bar{y}^* = \frac{(1-e^{-\delta})\left(V^{\frac{1}{c}}-1\right)}{b(1-e^{-(d+\delta)})}$	$\frac{\boldsymbol{a}e^{-d}(1-e^{-\delta})}{(1-e^{-d})(1-e^{-(d+\delta)})}$
(4)	$\bar{x}^* = \frac{(1-e^{-d})}{e^{-d}(1-e^{-(d+\delta)})}\left[\frac{V-1}{q}\right]^{\frac{1}{n}}$ $\bar{y}^* = \frac{(1-e^{-\delta})}{(1-e^{-(d+\delta)})}\left[\frac{V-1}{q}\right]^{\frac{1}{n}}$	$V = \frac{\boldsymbol{k}e^{-d}(1-e^{-\delta})}{(1-e^{-d})(1-e^{-(d+\delta)})}$
(5)	$\bar{x}^* = \frac{A(1-e^{-d})}{ce^{-d}[(1-e^{-(d+\delta)})]}\left[\frac{e^{-d}}{V}-1\right]^{-1}$ $\bar{y}^* = \frac{A(1-e^{-\delta})}{c[(1-e^{-(d+\delta)})]}\left[\frac{e^{-d}}{V}-1\right]^{-1}$	$V = \frac{\boldsymbol{c}e^{-2d}(1-e^{-\delta})}{(1-e^{-d})(1-e^{-(d+\delta)})}$
(6)	$\bar{x}^* = \frac{(lnV)(1-e^{-d})}{e^{-d}(1-e^{-(d+\delta)})}$ $\bar{y}^* = \frac{(lnV)(1-e^{-\delta})}{(1-e^{-(d+\delta)})}$	$V = \frac{\boldsymbol{q}e^{-d}(1-e^{-\delta})}{(1-e^{-d})(1-e^{-(d+\delta)})}$
(7)	$\bar{x}^* = \frac{k\sqrt{V-1}(1-e^{-d})}{e^{-d}(1-e^{-(d+\delta)})}$ $\bar{x}^* = \frac{k\sqrt{V-1}(1-e^{-\delta})}{(1-e^{-(d+\delta)})}$	$V = \frac{\boldsymbol{c}e^{-d}(1-e^{-\delta})}{(1-e^{-d})(1-e^{-(d+\delta)})}$

As bifurcation parameters are increased further, it leads to a series of period-doubling bifurcations, wherein a cycle of period 2^k loses stability and a stable cycle of period 2^{k+1} is born and eventually leads to chaotic dynamics.

III. RESULTS AND DISCUSSION

In the following, we show that the solutions of system with (3) behaves as given in Fig. 1.

When $a < a_0$ equilibrium $\bar{E}_0(0,0)$ is stable and when a is increased trajectoris approach the periodic solution with period 1 as shown in Fig. 2.

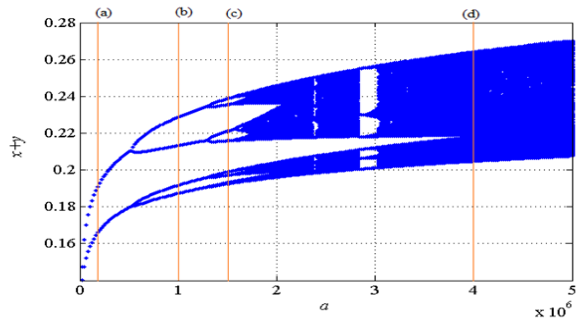


Fig1.

Bifurcation diagram with equation (3) with $a = 100,\ d = 0.2,\ \delta = 0.4, c = 100\ and\ \ b = 1.$

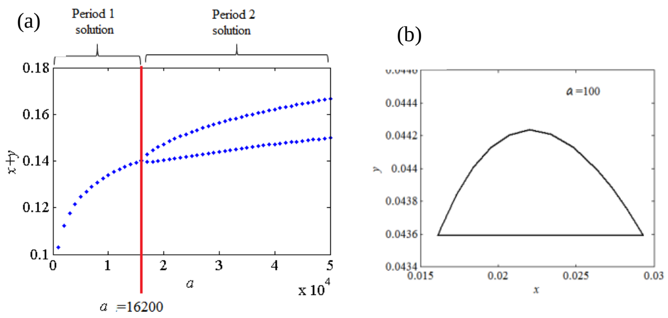
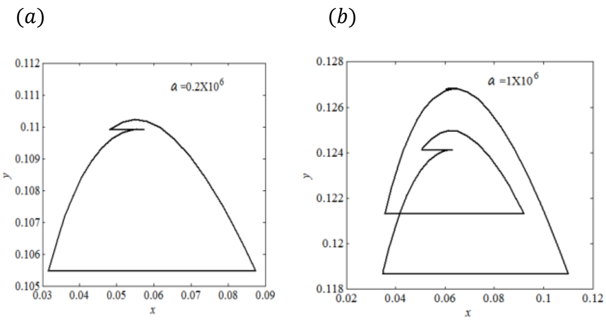


Fig.2. (a) Bifurcation diagram until period 2 solution which starts at a=16200. (b) Periodic coexistence of immature and the mature population with $a = 100, d = 0.2, \delta = 0.4, c = 100, b = 1$Period 1 solution.

The resulting bifurcation diagram in Fig. 2(a) clearly shows that first period-doubling is at $a = 16200$. Corresponding to the bifurcation diagrams Fig. 3 illustrates the relationships shows that birth pulse provides a natural period or cyclicity that allows for a period-doubling route to chaos. Increasing $a$leads to a cascade of period-doubling bifurcations and finally to the appearance of chaotic strange attractors.



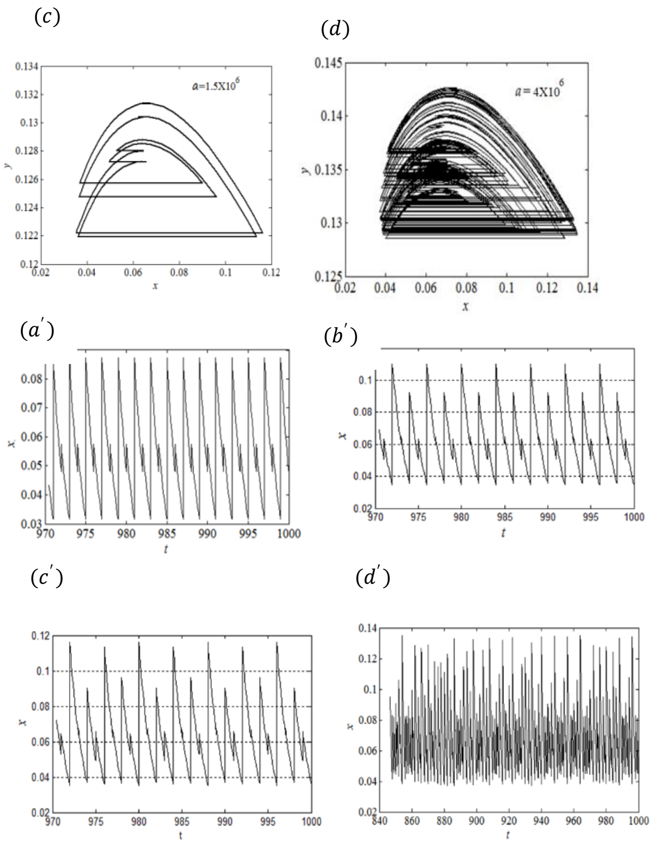
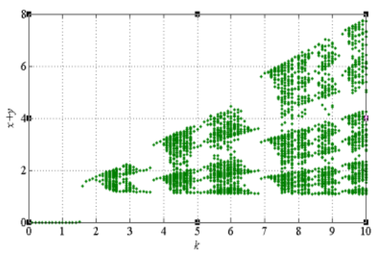


Fig 4: Period-doubling cascade to chaos (a) a 2-period solution (b) a 4-period solution (c) a 8-period solution (d) a strange attractor (a')Time series for 2-period solution of the immature population (b')Time series for 4-period solution of the immature population (c')Time series for 8-period solution of the immature population (d')Time series for strange attractor period solution of the immature population with $a = 100, d = 0.2, \delta = 0.4$ $, c = 100, b = 1$.



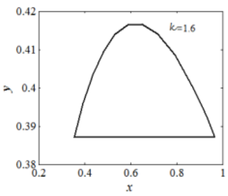


Fig. 4 Bifurcation diagram of equation (4) with $d = 0.6, \delta = 0.4$, $n = {}^{1}/_{14}$ *and* $q = 1$(With Shepherd Function)

Similarly, as shown in the bifurcation diagram of figure 4 and 5, increasing k and q leads to a cascade of period-doubling bifurcations and finally to the appearance of chaotic strange attractors.

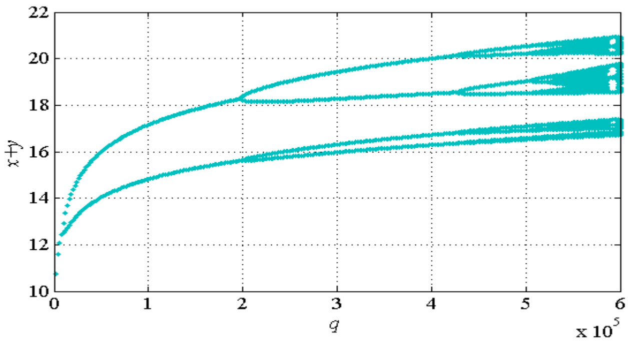


Fig 5. Bifurcation diagram with (6) with $d = 0.2,\ \delta = 0.4$

IV. Discussions and Conclusion

The main objective of the paper is to study the dynamic behavior of a stage structured single-species population model for different birth pulses. We considered five different birth-rate functions of which are functions of population density and we have performed this analysis for some functions [7]. Then we analyse the relationships between the differential dynamical system with birth pulses and the discrete dynamical system from which we obtain an exact periodic solution of systems which are with only by special case of birth function and obtain the threshold conditions for their stability. Once this threshold value exceed, it passes through a series of period doubling bifurcation that eventually lead to chaotic attractors. We have shown bifurcation diagrams to analyse the dynamic behavior of discrete dynamical systems. It is shown that dynamical behaviors of models with birth pulses are very complex. It is related to the fact that minor changes in parameter or initial values can strikingly change the dynamic behaviors of the system. We conclude that for different birth pulses, in effect, provide a natural periods that allow for a period-doubling routes to chaos.

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